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:: The Idea Behind Arbitrage Pricing ::

In my opinion, one of the concepts that new starters in quantitative finance find most difficult to grasp is the idea of valuing products through the elimination of arbitrage (**arbitrage pricing techniques**). This concept is fundamental to the pricing of all derivative products and the purpose of this article is to give the reader an introduction to the mechanism. It is natural for a new starter to use **expectation pricing** whereby the price of an asset today (for instance a derivative) is related to the expected future cash flows that the asset will generate. Although intuitive, this is not the method employed in practice. Such pricing mechanisms generate *arbitrage* opportunities in the market and the consensus is that any acceptable pricing mechanism should not do this.

In order to get started,

Definition 1: Arbitrage (1) - An opportunity in the market to realize *risk-free* profits.

As an introduction to this idea, consider a stock that has a dual listing (i.e. listed in 2 exchanges). If the quoted price on exchange A is higher than that on exchange B (after accounting for the foreign exchange differential and transaction costs involved in trading either product) then a trader can lock into risk less profits by buying the stock on exchange A and selling it immediately on exchange B.

This is the easiest description for an arbitrage; an opportunity to make a guaranteed profit. A more formal definition (and one that will be used explicitly in this article) is the following.

Definition 2: Arbitrage (2) - An opportunity to set up a portfolio at zero cost, but which is *guaranteed* to either increase in value or remain at zero. Mathematically, this implies:

$V(0) = 0$ condition 1

with,

$P(V(t) \geq 0) = \text{AND } P(V(t) > 0) > 0$ condition 2

where $V(t)$ implies the value or worth of the portfolio at time 't' and P( ) implies the probability of an event in the brackets.

This is a more formal definition of arbitrage, but implies the same concept. It defines arbitrage as the possibility of generating capital, with no possibility of making a loss (condition 2) even when no risk is taken (condition 1). This is also described as an opportunity to make something out of nothing (sometimes referred to as a *free lunch*).

The importance of these definitions is now considered via example. In particular the following points are addressed:

- How does one consider arbitrage when pricing a contract?
- Why does the expectation pricing technique not work in general?

Example 1: Consider a **forward contract** on a stock S.

Definition 3: Forward Contract - An agreement between 2 parties, whereby the seller agrees to deliver (to the buyer) one unit of stock A at some future date (the *maturity* of the contract) at some price which is decided today. Payment is upon delivery (i.e. at the maturity of the forward contract). Note that this is an agreement between the 2 parties, which must be realized.

A forward is a simple derivative security, since its value is derived from another security (in this case stock S). The problem that is now considered is the pricing of the forward of maturity T. In the arbitrage-pricing framework this is done according to the following argument. The *fair* or *arbitrage* price of the forward is the price that would not allow another party to set up a portfolio in the forward (and any other products) which will allow them to realize an arbitrage (i.e. risk less) profit.

For the purposes of the example, the asset is assumed to follow the following **stochastic differential equation**:

$$\delta S = \mu \delta t + \sigma dX$$

'S' being the stock price, μ the return from the stock, σ is the stock volatility, representing an incremental increase and X is a noise factor, which makes the behavior of the stock price random. It can not be

determined in advanced and is only revealed at time t. For the purposes of the work at hand the randomness is modeled by Brownian motion (see earlier article, '**What is a simple random walk?**')

Using this view for the behavior of the movement of the stock price, the price at a future time T of the stock is *expected* to be:

$$S_T = S_0 + \mu T$$

since the *expectation* of any Brownian motional increment is zero (**see earlier article**). In terms of pricing a forward of maturity T, then, one will expect the value S_T to play a very significant part. It is after all the expected future value of the asset.

For the moment this attitude to pricing will be accepted (it is wrong, but the reasons for this are discussed below). The value of the asset at time zero (e.g. today) is S_0 and it is expected to hit ($S_0 + \mu T$) by time T . Hence, the forward contract is priced at this value, meaning that at time T, the seller will surrender one unit of stock S to the buyer at a price of ($S_0 + \mu T$) , payment being made at time T.

Before continuing, another definition:

Definition 4: Risk free interest rate (r) - The (non-random) risk free rate of growth of capital. An initial sum X invested at r for a period t grows to $X(rT+1)$ with certainty. For a more complete account refer to

earlier article ('**Continuous compounding interest rates**').

Returning to the example that was developed above, consider a seller of the forward contract that exercises the following strategy at time zero:

Agree term of forward contract with seller (i.e. to sell at time T one unit of stock S for a sum of ($S_0 + \mu T$) defined above).
Borrow (go short) cash to the value of S_0 at the risk free rate.
Use the borrowed cash to purchase (go long) one unit of stock S from the market (which costs S_0 since the seller is at time zero) and hold until time T.

The strategy leaves the seller with a net profit of $S_0.(\mu - r)T$. Furthermore, this cash flow is guaranteed and it is risk-free. After time zero the seller only had to sit back and wait for time T to realize the profit. Hence, the forward price calculated earlier ($S_0 + \mu T$) has generated an arbitrage opportunity in the market. It is from this consideration that the arbitrage pricing argument extends.

Had the value of μ been less than the value of r then with the present strategy the seller is assured right from time zero to realize a loss at time T. However, under these circumstances the buyer of the forward contract will realize the arbitrage profit.

The *fair* price for any derivative is one that will not generate *any* arbitrage opportunity in the market. It therefore must completely discount what *may* happen to the stock price in the future and consider the prevention of an arbitrage opportunity, i.e. price the forward at ($S_0 + \mu T$).

It is now hopefully apparent that the expectation pricing technique will not work in general (unless μ = r miraculously). This demonstrates that even if there is a strong belief that the stock at time T will be worth ($S_0 + \mu T$) > ($S_0 + rT$), at time T but this does not correspond to an arbitrage profit. The investors sentiment towards the value of S at time T means that he is taking a risk. There is no guarantee that the stock will reach $S_0 + \mu T$. The SDE that was assumed earlier is only a vision. It may well be a great model to describe the behavior of the stock price but it is not a guarantee. The stock market could crash tomorrow, in which case the investor loses out. This is a risk (no matter how small) and the investor should expect to be rewarded for it if his vision of the stock price proves correct. The arbitrage pricing mechanism has only concentrated on preventing the generation of arbitrage opportunities. Opportunities such as the one described in this last paragraph remain part of the gamblers' world.

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