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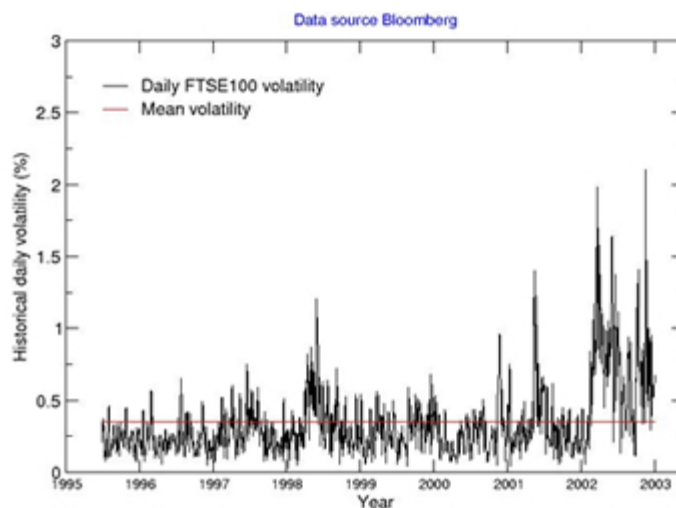
:: FUNDAMENTALS

:: ARCH and GARCH models for forecasting volatility ::

In the **Black-Scholes model**, the **volatility** of the underlying asset is the only non-directly observable variable. For this reason, it is necessary to devise some method where by one can estimate (efficiently) and possibly anticipate the volatility. In a previous article, we have already discussed a rather popular method of deducing the **implied volatility** from the Black-Scholes model and option prices. This method has an advantage over direct estimations based on historical price changes, since it reflects how much volatility the market currently assumes within the Black-Scholes framework. Quite often, the implied volatility is found to give rise to a *skew*, *smile* or *frown*, (as **previously illustrated**) depending upon the asset or market.

One problem with using the implied volatility is that whilst it takes into account the current view of the market, it does not give us any insight into possible future changes in volatility. Given that the value of an option is primarily driven by the volatility, making predictions is a valuable tool from a practitioner's perspective. To achieve this, we turn back our attention to historical price movements.

In the figure below, we have calculated the daily percentage volatility rate (per day) of the FTSE 100 over several years (data provided courtesy of **Bloomberg**). [Note, for an index such as the FTSE 100, the volatility rates are generally less than that of a single stock option.]



We see that the volatility rates are higher over certain periods and lower in others. The most striking feature is that periods of high volatility tend to *cluster* together. Therefore, one would expect the volatilities to be correlated to some extent. The other noticeable feature is that the

volatility tends to revert to some long-running average - a property commonly known as *mean-reversion*. The flat **red** line in the figure above shows the average volatility over the entire period. The mean-reversion nature of the volatilities helps ensure that the process remains statistically stationary. [A discussion into stationary Vs non-stationary stochastic processes is beyond the scope of this article. It is common to adopt a *weaker* definition of a stationary process where the first two order moments remain finite in time.]

Autocorrelation structures. We take a slight detour to introduce the definition of the *autocorrelation function*. The *correlation function* between two time series X and Y is given by the expression

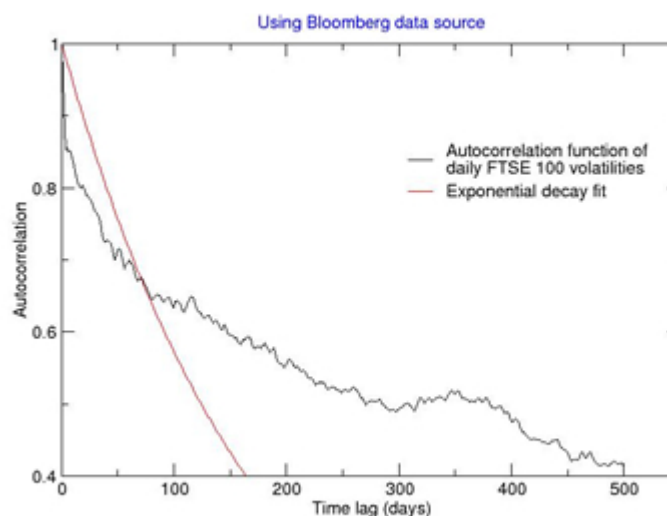
$$C(X, Y) = \frac{\langle (X - \mu_X) (Y - \mu_Y) \rangle}{\sigma_X \sigma_Y}$$

where μ_X (μ_Y) and σ_X (σ_Y) are the mean and variance estimates of X (Y) respectively, and $\langle \dots \rangle$ denotes the mean value of the expression inside the brackets. The autocorrelation function is calculated by setting $Y = X(t + \delta)$, where δ is some forward time lag of the time series X. Hence, the autocorrelation function may be expressed as

$$C(\delta) = \frac{\langle [X(t) X(t + \delta)] \rangle - \mu_X^2}{\sigma_X^2}.$$

The autocorrelation function is an *average* measure of the correlations that exist within a time series.

To demonstrate the correlated nature of the volatilities, we have calculated the autocorrelation function of the daily volatility rates, illustrated in the figure below.



One immediately sees that the volatility autocorrelation decays extremely slowly with increasing time lag. The form of this volatility autocorrelation has been empirically suggested to be either exponentially decaying, or exhibiting long-range memory (power-law decay). We have performed a least squares estimate (**red** line) for an exponentially decaying function of

the form

$$y = \exp(-ax)$$

where a is a constant, as a rough comparison to the empirical data. The exponential decay fit appears to be inadequate for characterising the form of the autocorrelation decay.

ARCH models

The *autoregressive conditional heteroskedasticity* model was introduced by Engle (1982) to model the volatility of UK inflation.

As the name suggests, the model has the following properties:

- 1) Autoregression - Uses previous estimates of volatility to calculate subsequent (future) values. Hence volatility values are closely related.
- 2) Heteroskedasticity - The probability distributions of the volatility varies with the current value.

In order to introduce ARCH processes, let us assume that we have a time series of asset price quotes P_i for each time step i . We calculate the fractional change in the price of the asset between time step i and $i+1$ using

$$x_i = \frac{P_i - P_{i-1}}{P_{i-1}}.$$

Furthermore, we are required to determine the long-running historical volatility (e.g. over several years) denoted by S_{long} . In the first figure above, S_{long} is illustrated by the flat **red** line. We have seen that the volatility rates fluctuate about this mean long-running mean volatility, therefore, it seems reasonable to incorporate this quantity in the ARCH model.

Formally, an ARCH(m) process may be expressed mathematically as

$$\sigma_n^2 = \gamma S_{long} + \sum_{i=1}^m \alpha_i x_{n-i}^2$$

where σ_n is the volatility at the n^{th} time step, γ and α_i are weighting factors that satisfy

$$\gamma + \sum_{i=1}^m \alpha_i = 1.$$

Here m denotes the number of observations of x_{n-i} used to determine σ_n . The most common ARCH(m) process used to model asset price volatility dynamics is the ARCH(1) model where

$$\sigma_n^2 = \gamma S_{long} + \alpha x_{n-1}^2,$$

or

$$\sigma_n^2 = \gamma S_{long} + (1 - \gamma) x_{n-1}^2,$$

using the above relation.

GARCH models

Bollerslev (1986) later proposed a more generalised form of the ARCH(m) model appropriately termed the GARCH(p,q) (General-ARCH) model. The GARCH(p,q) model may be written as

$$\sigma_n^2 = \gamma S_{long} + \sum_{i=1}^p \alpha_i x_{n-i}^2 + \sum_{j=1}^q \beta_j \sigma_{n-j}^2.$$

The p and q denote the number of past observations of x_{n-j} and σ_{n-j} , respectively, used to estimate σ_n .

The EWMA model

The Exponentially Weighted Moving Average model (EWMA) is a special case of the GARCH(1,1) model where $\gamma=0$. Thus,

$$\sigma_n^2 = \alpha x_{n-1}^2 + \beta \sigma_{n-1}^2.$$

Since $\alpha + \beta = 1$, we may express the EWMA model as

$$\sigma_n^2 = \alpha x_{n-1}^2 + (1 - \alpha) \sigma_{n-1}^2.$$

The EWMA model differs from ARCH and GARCH models since it does *not* mean-revert. The preference between these different models is dependent upon many factors. For example, the asset class, forecasting time frame under consideration, and the efficiency with which the weighting parameters may be calibrated to the time series. Whilst the *maximum likelihood estimators* method may be the most obvious method to select for calibration with empirical data, more efficient algorithms have also been put forward.

Since these volatility forecasting models were introduced, there have been many alternatives/modifications proposed to these models to better their use in volatility forecasting.

Written by Alessio Farhadi.

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