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Deriving the Black-Scholes Equation

Assuming that the movement of a share price follows the geometric Brownian motion given by

$$dS = \mu S\,dt + \sigma S\,dX$$

The value *V* of an option or other derivatives contingent on *S* is a function of *S* and *t*, then from **Itô's Lemma** (see **article**), we get

$$dV = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S}\, dX$$

Now consider a portfolio of value Π constructed by longing one option and shorting Δ amount of shares, this gives

$$\Pi = V - \Delta S$$

Differentiating this gives the change in portfolio as

$$\begin{aligned} d\Pi &= dV - \Delta\, dS \\ &= \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \\ &\quad \sigma S \frac{\partial V}{\partial S}\, dX - \Delta\, dS \\ &= \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} - \mu S \Delta + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \\ &\quad \left(\frac{\partial V}{\partial S} - \Delta \right) \sigma S\, dX \end{aligned}$$

Assuming that there are no arbitrage opportunities, the value of Δ can be chosen such that the portfolio will earn the same rate of return as other risk-free securities,i.e.

$$d\Pi_{risk\text{-}free} = r\,\Pi\,dt$$

Where r is the interest rate. Equating the change in portfolio to the risk-free return of Π, i.e. **hedging** the portfolio, gives

$$\begin{aligned} d\Pi &= d\Pi_{risk\text{-}free} \\ &\left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} - \mu S \Delta + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \\ &\quad \left(\frac{\partial V}{\partial S} - \Delta \right) \sigma S\, dX = r\,\Pi\,dt \end{aligned}$$

By equating the change in portfolio to the risk-free return, we have eliminated the randomness associated with the portfolio, this is only true if the coefficient of dX is equal to zero, therefore

$$\Delta = \frac{\partial V}{\partial S}$$

So this gives

$$\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} - \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - r\Pi = 0$$

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - r \left(V - S \frac{\partial V}{\partial S} \right) = 0$$

Rearranging gives

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

This is the **Black-Scholes equation** for pricing options.

In deriving this equation, we assumed that:
1. There are no arbitrage opportunities (no free lunch).
2. Short selling of shares is possible at all times.
3. No transaction costs or taxes in setting up a portfolio.
4. All securities are perfectly divisible.
5. Trading can take place continuously.
6. The underlying share pays no dividends during the lifetime of the option.
7. The risk-free rate *r* and the share volatility σ are known over the lifetime of the option.

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