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:: FUNDAMENTALS

:: Interest Rate Modelling 2 - Short Rate Models ::

Following on from an [earlier article](#), here we describe the first generation of interest rate modelling. This approach models the interest rate (or short rate) directly. In general, the short rate is assumed to follow some process:

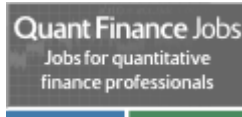
[\[Graphics/Images/interestrat](#)

The bond pricing PDE, which can be found in several texts, is then:

[\[Graphics/Images/interestrate2_gr_2.gif](#)

The purpose of this article is not to address the mathematics and the arguments that lead to the derivation or solution of the PDE above. Instead, the emphasis is shifted onto the logic and intuition behind fixed income derivatives. You can summarise interest rate modelling by saying that it is all about finding a form/expression for $u(t, r)$ and $w(t, r)$ above. λ is defined as the 'market price of risk' which brings us to the first unusual feature of the Fixed Income markets. The PDE above is derived using the usual delta hedging arguments, but applied to bonds. A bond is treated as a derivative of interest rate; much like an option is a derivative of the underlying stock. The difference however, is that the stock can be traded openly on the market. Interest rate is not an asset, it is more like a market parameter, and can not be traded. Hence, the standard Black-Scholes argument (i.e. to hedge the derivative with the underlying), breaks down. The only instrument that can be used to hedge a bond is another interest rate derivative; another bond of perhaps longer maturity than the bond that we are trying to price. But this seems counter-intuitive. How can you price an interest rate instrument (e.g. bond or bond option) using another instrument (bond) which you do not know how to price in the first place? This is kind of like a 'chicken and egg' situation, and is why you need a market price of risk. The market price of risk introduces a basis; a start point. The argument is as follows. Forget about the 'chicken and egg' situation, and assume that the current market of bonds is perfect. Therefore, assume that the prices of bonds (driven by demand and supply) are arbitrage free, so that the instrument that you are hedging with is well defined and not an unknown. The model and PDE has to incorporate this 'basis', so we use the market price of risk to force the model to accept it. In short, use market price of risk as a calibration parameter to fit the model to the market (which we assume is perfect and arbitrage free), and to ensure that the values of instruments output by the model/PDE closely resemble the actual market prices of those same instruments.

However, there is no reason why the assumption above should hold. Many people believe the market is wrong and feel that they have a better understanding of what bond prices should be. Recently there has been an



:: Collaborations ::



explosion of such arbitrageurs, and many have been quite successful at it (the hedge fund Long Term Capital Management is no exception, although it eventually went down). However, the very existence of such arbitrageurs might worry us, in that we are essentially calibrating to an imperfect market, and therefore generating imperfect/non-arbitrage free prices. But it should not worry us too much because such players push the market to become more efficient and more arbitrage free. In practice, calibration is the best and most widely adopted way forward and the assumption that the current market is arbitrage free prevails.

So how does the calibration occur in practice? There are two approaches which we can take. The first is to solve the PDE above for bonds which are already priced in the market. The only unknown parameter is θ ; therefore this parameter can be calculated or fitted to the market prices via some sort of error minimisation (minimisation of the absolute error between the model price and market price). For e.g. if the market trades 10 different bonds issued by the same corporate, then there is no reason why θ calculated from all the bonds will agree. This hints to an inconsistency or arbitrage in the market, but the fundamental assumption will prevail (that being that the market is in fact efficient) and we attempt to fit to all the 10 bonds simultaneously by making θ a function of time if necessary. This is the calibration step, but is also called yield curve fitting because you are essentially fitting the model to the yields on the different bonds (recall the [yield/price relationships](#)).

The second approach is to just incorporate θ into the expression for $u(t,r)$ and calibrate the whole term together. Taking this a step further, we could in fact also make w into a calibration parameter. The reason this is done is because a greater number of market instruments can be calibrated to if you have more parameters to achieve this fit. So instead of solving the PDE for just bonds, we solve it for Bond options (caps and floors) and try to calculate $u(t,r)$ and $w(t,r)$ to fit all of these instruments. The optimisation becomes more intense for sure, but at least you are consistent with a broader market. And in practice, this is what happens most frequently. The θ is dropped and all the parameters are used in the calibration.

Note that sometimes a closed form for the derivative exists (be it a bond, or bond option, etc.), but sometimes the calibration is numerical. This depends on the forms for $u(t,r)$ and $w(t,r)$. But the key is that once the parameters have been deduced, other derivatives can be priced. The same PDE is still valid for bond options, coupon bonds, etc., but the boundary conditions change.

You can show that the solving of the PDE above is equivalent to calculating the expected discounted payoff of the derivative security (whether bond, or bond option) but under the following dynamics for the short rate:

$$dr = \theta(r,t)dt + \sigma(r,t)dW$$

where,

$$\theta(r,t) = \frac{\partial \ln P(r,t,T)}{\partial t}$$

The mathematics to show this are omitted, but it is basically an

application of a famous theorem that links PDEs to expectations: The Feynman-Kac theorem. Many people talk about the dynamics of the interest rate as now being the 'risk neutral' dynamics, because of the analogy to the Black-Scholes model. But this is misleading. Risk neutral in the Black Scholes setting came about as a result of perfect hedging, whereas here we have described that perfect hedging is not possible, and that there is always an element of risk: that risk being that you are calibrating to a market which may not be arbitrage free. However, it has become conventional to speak about the risk neutral dynamics, and this aside, the use of the risk neutral dynamics is a powerful concept, so much so that the most famous short rate models were postulated as being under the risk neutral dynamics (see below). Once θ and w are known ANY derivative can be priced as the expected future payoff (whether by Monte Carlo or trees), discounted at the risk free rate described by the dynamics above.

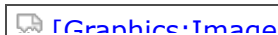
Some short rate models

The simplest model to introduce is the Ho-Lee model for interest rate:



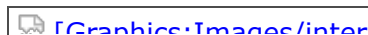
The postulate is that this is the risk neutral process so that θ includes the market price of risk term. But before even worrying about that you should realize that the model is not very realistic because interest rates do not exhibit this type of growth behaviour. The reason why the model prevailed is because the mathematics involved in the calibration is relatively easy: a closed form solution for bond prices and bond options exists and calibration can be made without any numerical computation (see Wilmott for example).

To circumvent the weaknesses of the Ho-Lee model, the Vasicek model was introduced:



This, for the first time, treated interest rate as a mean reverting process.

Clearly when $\theta = 0$ the interest rate has no drift, so that the ratio $\frac{r(t)}{\theta}$ essentially describes the equilibrium interest rate, or the mean interest rate about which fluctuations occur. The parameter θ is the mean reversion rate which is a measure of how fast the interest rate is pulled back to the long term level. Whilst still tractable, the weakness here was that a good fit could not be obtained to the yield curve. But it was soon observed that if the parameters were allowed to be time dependent, a better calibration could be obtained. The Hull-White model is essentially the Vasicek model with time dependent parameters:



Again, these describe the risk neutral dynamics of interest rate, that is, the drift term is assumed to incorporate the market price of risk, so that the only way to extract an expression for the drift is via calibration. The Hull-White model was also a breakthrough because it can be discretised into a trinomial tree (which is used widely in interest rate markets as well as energy markets).

The weakness of the models above is that they allow interest rates to go negative (in all cases, if the contribution from the random component is large and negative then it could potentially pull the spot rate into negative territory). The next model was introduced to counter this specific problem. It is the Cox Ingersoll Ross model:



Since the volatility is a function of the interest rate, the interest rate can never fall below the level zero barrier; as this happens, the contribution from the random term above diminishes, while the drift term (which pulls the interest rate up towards the long term level) prevails. The mathematics for calibration/PDE solving associated with the CIR model becomes more complicated but the model is still tractable for certain values of ρ .

In summary, we have introduced the logic behind and need for the market price of risk. This is nothing more than a parameter that allows us to fix the model to the market. We have also described some of the popular interest rate models and highlighted their strengths and weaknesses and also introduced the idea of 'risk neutral interest rate dynamics'.

Written by Samy Mohammed

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