



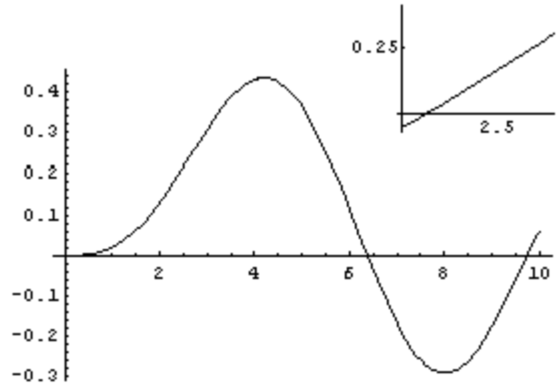
:: Itô's Lemma ::

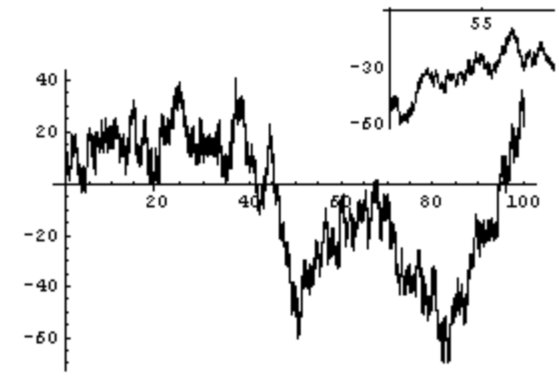
Sometimes a Stochastic process S can be split into the action of a random walk process (see **article**) acting on top of a deterministic drift $\mu(t)$. In the continuous limit the random walk becomes what is known as a **Wiener** process X(t). During an interval δt the process moves deterministically by $\mu(t)\delta t$ and randomly by $\sigma\delta X(t)$. This forms our basic example of a **Stochastic Differential Equation** (SDE).

$$\delta S(X(t), t) = \mu(t)\,\delta t + \sigma\,\delta X(t)$$

No matter how small an interval δt, there will always be a small contribution from the δX(t) term. In fact by zooming into the function S(t) one gets something that statistically looks the same (see second graph below). This occurs since the Wiener process has a fractal nature. This is very different from the **Ordinary Differential Equation** (ODE) where the Wiener term X(t) is absent. For the ODE we find that as long as $\mu(t)$ is a smooth function then our function S in the interval [t,t+δt) approaches a straight line as $\delta t \to 0$ (see first graph below).

:: Collaborations ::





Top graph: ODE. Bottom graph: SDE.

In **Ordinary calculus** if a function V depends on t and on x(t),the total derivative of V is found to be

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial t} dt$$

This is can be obtained in the limit $\delta t \to 0$ of Taylor expansion of V(x(t),t) by neglecting higher orders of δt. In this limit both $\partial_x V$ and $\partial_t V$ remain constant within the interval [t,t+δt), explaining why one ends up with a straight line by zooming into V(x,t).

Now consider V(X(t),t), if we Taylor expand this we obtain:

$$\delta V \approx \frac{\partial V}{\partial t}\delta t + \frac{\partial V}{\partial X}\delta X + \frac{1}{2}\left(\frac{\partial^2 V}{\partial t^2}(\delta t)^2 + 2\frac{\partial^2 V}{\partial t\,\partial X}\delta X\delta t + \frac{\partial^2 V}{\partial X^2}(\delta X)^2\right) + \ldots$$

By taking the limit $\delta t \to 0$ of its Taylor expansion we obtain

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial X}dX + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}(dX)^2$$
$$= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}\right)dt + \frac{\partial V}{\partial X}dX$$

This follows from the fact that $(dX)^2$ is of order dt, and is the statement in the continuos limit that for a simple random walk that the variance is proportional to the time. Also due to the rapid changing nature of X(t), neither $\partial V/\partial X$ nor $\partial^2 V/\partial X^2$ remain constant in the interval [t,t+δt) in this limit.

V may be a function of S and t. In this case our Taylor expansion yields

$$dV = \left(\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2}\right)dt + \sigma\frac{\partial V}{\partial S}dX$$

This is **Itô's lemma**. It relates the change in a stochastic function to the change in the stochastic variable.

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