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:: FUNDAMENTALS

:: Log-Normality & Finance ::

Consider the value of a share $S(t)$ of a certain company. An investor buying this share for $S(0)$ at time $t=0$ hopes that after a time T his investment does better than putting the same money into a bank at a risk free interest rate r . Thus:

$$\text{investor hopes : } S(T) \geq S(0) e^{rT}$$

Most companies have their share value outperforming the risk free interest rate or else people would not place their money in that company. Therefore the expected value of $S(t)$ should be something like

$$\overline{S(t)} = S(0) e^{\mu t}$$

where $\mu > r$.

How does $S(t)$ fluctuate?

Lets assume that the noise is additive **white noise**. This means that $S(t)$ changes deterministically by $\Delta S = r \Delta t$ and stochastically by $\Delta S = \sigma S \Delta X$ at each time interval. However, no matter how large Δt , as time evolves the fluctuations become increasingly more insignificant compared to the deterministic evolution. This means that as time goes on we are to expect a near to riskless return rate of r which we have assumed to be greater than r .

If this is the case no one would invest into a bank giving a risk free rate $r(t)$ - at least not for long term investments. It is obvious that this dynamics therefore cannot represent the evolution of S .

To avoid this we assume that the importance of the fluctuations with respect to the deterministic evolution remains constant over time. This means that the fluctuation is linear multiplicative white noise. The Langevin equation is thus:

$$dS = r S dt + \sigma S dX$$

We solve this using **Ito's Lemma**:

$$F(S, t) = \log S$$

$$dF = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial S} dS + \frac{1}{2} \frac{\partial^2 F}{\partial S^2} (\sigma S)^2 dt$$

$$= \mu dt + \sigma dX + \frac{\sigma^2}{2} dt$$

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$$S(t) = S(t_0) \exp \left\{ \sigma [X(t) - X(t_0)] + \left(\mu - \frac{\sigma^2}{2} \right) (t - t_0) \right\}$$

$$Y = \log \left(\frac{S(t)}{S(t_0)} \right)$$

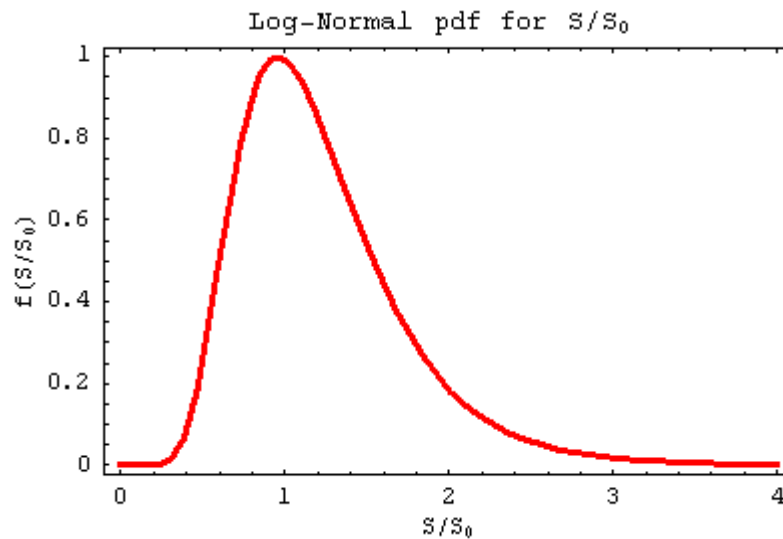
$$= \left(\mu - \frac{\sigma^2}{2} \right) (t - t_0) + \sigma [X(t) - X(t_0)]$$

Since $X(t) - X(t_0)$ has a Gaussian or Normal distribution of mean 0 and variance $(t - t_0)$, denoted as $\mathcal{N}[0, (t - t_0)]$, the distribution of Y is also Gaussian of the form :

$$Y \sim \mathcal{N} \left[\left(\mu - \frac{\sigma^2}{2} \right) (t - t_0), \sigma^2 (t - t_0) \right]$$

We thus say that $\frac{S(t)}{S(t_0)}$ follows a **Log- Normal** distribution.

Example: say $\sigma^2 = 0.4$, $\mu = 0.16$ and $(t - t_0) = 1$ then the probability distribution of $\frac{S(t)}{S(t_0)}$ looks as follows:



Martingales and Change in Probability Measure

Consider the process S that follows a **simple random walk** with $p=0.75$ and $q=0.25$. Is this a fair game? - Obviously not and therefore the stochastic process S in the continuous limit is not a Martingale. However, in the continuous limit this process can be modelled by the following stochastic equation:

$$\mu = 0.75 \times 1 + 0.25 \times (-1) = 0.5$$

$$\sigma^2 = 0.75 \times 1^2 + 0.25 \times (-1)^2 - 0.5^2 = 0.75$$

This representation can be done in the continuous limit since the fluctuations of the original process are Gaussian and only the first two

moments are important. This means that by knowing μ and σ we can mimic the evolution of S using a fair game Gaussian generated random variables dX . So can we transform the process S to a martingale?

Let dW be a second Gaussian distributed random number such that:

$$dX = dW + \gamma dt$$

Obviously the process dW is not a fair game with respect to the process dX . However if we consider an origin shift of γ - then this process becomes a fair game with respect to this new coordinate system. What we mean is that dW is a martingale under a new probability measure. This new probability measure is trivially obtained from the dX probability measure by a simple coordinate transformation. For every outcome in dX there is a unique outcome in dW and vice-versa (i.e. the mapping is one-to-one and onto). So substituting this in to the expression for dS we have:

$$dS = \mu dt + \sigma(dW + \gamma dt)$$

By choosing  [Gr] we get that

$$dS = \sigma dW$$

and thus dS becomes a martingale under this new transformed probability measure. Basically all this means is that dS is a fair game according to the process that follows the deterministic evolution .

This argument can be carried out also for the **Log-Normal process** whereby it is dS/S that is a martingale with respect to the process dW :

 [Graphi

i.e. dS is a fair game according to the process that follows the deterministic evolution . It thus called an exponential Martingale.

In Finance - is dS really a fair game?

No! - If $S(t)$ represent the value of an asset at time t , then the ratio  is a ratio of the expected return to the risk. However a true measure of the expected return is only obtained once we discount the risk free interest rate. So if we change our probability measure transformation to:

 [Graphic

then the process dS can be considered a fair game

In Finance this ratio is called the *market price of risk* - it reflects the risk-return tradeoff or better, the additional expected return necessary to induce investors to assume risk.

Note: 'Girsanov's Theorem' applied to Finance' gives a more mathematically rigorous discussion of the above.

Written by Raffaello Vardavas.

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