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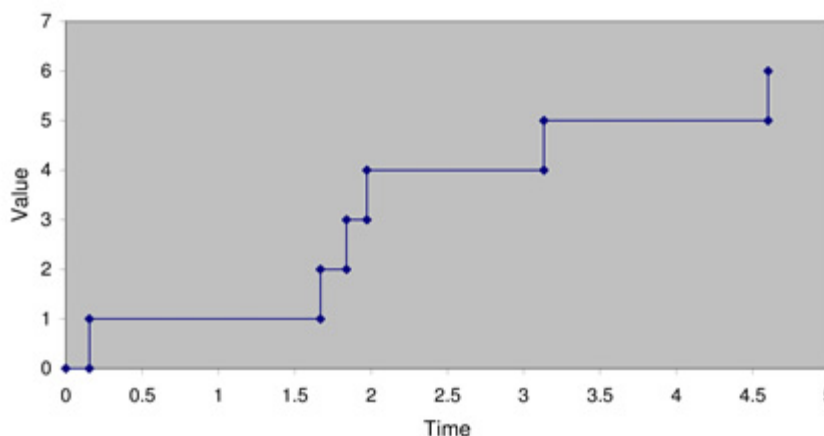
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:: FUNDAMENTALS

:: What is a Poisson Process? ::

A Poisson process is a pure jump process: a process that changes instantaneously from one value to another at random times. The following is a simulation of a standard Poisson process (where the jump sizes are restricted to 1).

Simulation of a Standard Poisson Process



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The model for such a process extends from the discrete time Poisson distribution. This states that the number of (Poissonly distributed) events (N) in a time interval $(0, T]$ is distributed according to:

$$P(N = x) = \frac{(\lambda T)^x e^{-\lambda T}}{x!} \quad \text{for } x = 1, 2, \dots, \infty$$

Here, the number of events corresponds to the number of jumps and λ is the intensity of the Poisson process; a measure of the 'frequency' of jumps, often scaled to units of per unit time. The Poisson process is a discrete probability distribution and has been successfully used to model the arrival times of certain events, or the occurrences of certain events, over a pre-defined period. The difference from most discrete distributions is that the number of occurrences can in theory (and with non-zero probability) be infinite.

Of particular relevance to finance (default modelling) is the waiting time between the arrivals of each event/jump. This is given by the exponential distribution and we will often be interested in the first arrival time τ .

Although the Poisson distribution is a discrete one, the inter-arrival and first arrival times are continuous exponentially distributed random variables. The PDF is:

$$p(\tau) = \lambda e^{-\lambda \tau}$$

The probability of the first jump occurring in the time interval $(0, s]$ is

then:

$$p(\tau < s) = \int_0^s \lambda e^{-\lambda t} dt = 1 - e^{-\lambda s}$$

It is important to note that the Poisson process is consistent with the Poisson distribution for,

$$P(N = 0) = \frac{(\lambda T)^0 e^{-\lambda T}}{0!} = e^{-\lambda T} = p(\tau > T)$$

We will mainly use in future articles the Poisson process from the exponential distribution perspective, i.e. the arrival of the first jump to characterise events like corporate default. But the Poisson process is also becoming increasingly important in modelling fat tailed processes, and underlying which exhibit jump diffusion (e.g. energy).

An important property of the Poisson process is the Markov property. Stated briefly, this is the 'loss in memory' property where the distribution of the Poisson process in the future is independent of the past. For e.g. at time 0 the probability of not observing a jump over a time horizon T is simply $\exp(-\lambda T)$ from the derivation above. Now assume that we return to the process after a time s and the process is still at 0 (i.e. no jump has yet occurred). The probability of not observing a jump for a further time T (i.e. no jump until time $T+s$) given that no jump has occurred until time s is:

$$P(\tau > T+s \mid \tau > s) = \frac{P(\tau > T+s)}{P(\tau > s)} = e^{-\lambda T}$$

Using the expressions derived earlier this probability is just the same as the time 0 probability of not observing a jump over a time horizon T . This illustrates the Markov property; the fact that the process has not jumped until time s (whatever s might be) does not dictate the probability of future jumps.

How is a Poisson process mathematically characterised? Quite simply, the value of a standard Poisson process after a time T has elapsed is simply:

$$N(T) = N(0) + \sum_{s < T} [N(s) - N(s-)]$$

The expression looks more daunting than it is. $N(0)$ is simply the initial condition (set to zero in a standard Poisson process). The latter term is the mathematical expression for 'the number of jumps in the time interval $(0, T]$ '. Since the process jumps finitely in infinitesimal time, the time s corresponds to an infinitesimal time step before time s , and where a jump is observed $[N(s) - N(s-)]$ is 1; otherwise it is 0. In its more useful form, the process can also be expressed as $dN(t)$ which models the change in the Poisson process over a time step dt . Using the Markov property the value of $dN(t)$ at any time t does not depend on the history of the Poisson process. Furthermore,

$$P(\tau < dt) = 1 - e^{-\lambda dt} = 1 - \left(1 - \lambda dt + \frac{1}{2} \lambda^2 (dt)^2 - \dots\right) \sim \lambda dt$$

because dt is very small. Thus, $dN(t)$ can be thought of as a random

variable that increases by 1 over a time step dt with probability λdt and is zero with probability $1 - \lambda dt$.

Jump Diffusion Calculus

The calculus of jump processes is somewhat tricky if you haven't seen it before. A Poisson process is not a continuous one and hence the ordinary rules of calculus do not apply. When a jump occurs, the value of the process shifts up from one value to another instantaneously. Hence the notion of derivative does not exist (it is infinite!), as the jump constitutes a discontinuity. If X is a stochastic diffusion process that can jump as well then it is called a jump diffusion:

$$dx = A(t, x) dt + B(t, x) dW + C(t, x) dN$$

The first two terms are the usual drift and white noise that have been used extensively to model stock prices in finance. The last term introduces the possibility of a jump occurring. 'dN' constitutes a standard Poisson process; over a time interval dt a jump of size 1 can be observed with probability dt . The scaling by $C(x, t)$ allows the jump size to vary.

Such models are becoming increasingly important in modelling stocks as they result in distributions with 'fatter tails' than the standard Ito processes. They are also being used to model energy and power prices where the jump behaviour is very often observed.

The key for mathematical finance is to now derive the SDE for a function $f(X)$. The key is to consider the process X as the sum of 2 processes; a continuous process:



and a pure jump process:



Then, to consider the Taylor series expansion of $F(x)$ by first considering the contribution from the continuous process and then the jump process:



The last term arises from the jump component. $[x + C(x, t)]$ denotes the value of the process x just after a jump. The majority of the times the last term is zero because $dN=0$. Only in those cases when a jump occurs the last term is non-zero and the jump in x is also observed in the function F .

Why is this important? Because it allows us to use an alternative form of Ito's Lemma to derive option prices on stocks that can jump. Very rarely can the jump be hedged.

Written by Samy Mohammed.

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