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:: HELP :: CONTACT

- :: FUNDAMENTALS
- :: PUBLICATIONS
- :: SOFTWARE & DATA
- :: BOOK REVIEWS
- :: JOB LISTINGS
- :: EVENT LISTINGS
- :: FORUMS
- :: EDUTAINMENT
- :: USEFUL LINKS
- :: ABOUT US

:: FUNDAMENTALS

:: The Simple Random Walk ::

Consider a person that is allowed to take either a step forward or backward depending on the outcome of a random event $Z = \{-1, +1\}$. At fixed time intervals he has a constant probability p to move forward ($Z = +1$) and a probability $q = 1 - p$ to move backward ($Z = -1$). During any one interval the expected displacement ($E(Z)$) and variance ($Var(Z)$) is:

First Moment:

$$E(Z) = (+1)p + (-1)q = p - q$$

Second Moment:

$$E(Z^2) = (+1)^2 p + (-1)^2 q = 1$$

Second Cental Moment:

$$\begin{aligned}\sigma_Z^2 &= Var(Z) = E(Z^2) - [E(Z)]^2 \\ &= 1 - (p - q)^2 = (p + q)^2 - (p - q)^2 \\ &= 4pq\end{aligned}$$

We are interested in knowing where the person will likely be with respect to his starting position, after he has taken n steps. We denote this displacement as X_n . The outcome of the random event Z has the Markov property. This simply means that the next step is independent from any of its previous outcomes. So this allows us to linearly scale up all the moments of the displacement by a factor of n . An important result that follows is that since the number of steps taken n , is proportional to the time duration, the variance of a random walk increases linearly with time.

$$X_n = \sum_{i=1}^n Z_i$$

$$E(X_n^m) = E\left(\sum_{i=1}^n Z_i^m\right) = \sum_{i=1}^n E(Z_i^m) \quad (\text{by Markov})$$

$$E(X_n) = n(p - q)$$

$$Var(X_n) = 4npq$$

What is his probability $p(X_n)$ for the person to be at position X_n steps in front of his initial position after he has taken these n steps? He could have reached this position by a number of different ways, but we know that for each, the total number of forward steps taken f minus the number of backward steps taken b must be equal to X_n (i.e. $X_n = f - b$). Let us consider just one of the possible ways he could have reached position X_n . The probability of doing so is $p^f q^b$. This is the same for all the different combinations of choosing f forward steps out of n total steps. So all we need to do multiply this probability by the total number of ways in reaching his final position.

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$$p(X_n) = \binom{f+b}{f} p^f q^b =$$

$$\binom{n}{(n+X_n)/2} p^{\frac{(n+X_n)}{2}} q^{\frac{(n-X_n)}{2}}$$

$$\binom{n}{m} = \frac{n!}{m! (n-m)!}$$

Recall that a binomial random variable has distribution:

$$B(n, m) = \binom{n}{m} p^m q^{n-m}$$

$$E(B) = np$$

$$\text{Var}(B) = npq$$

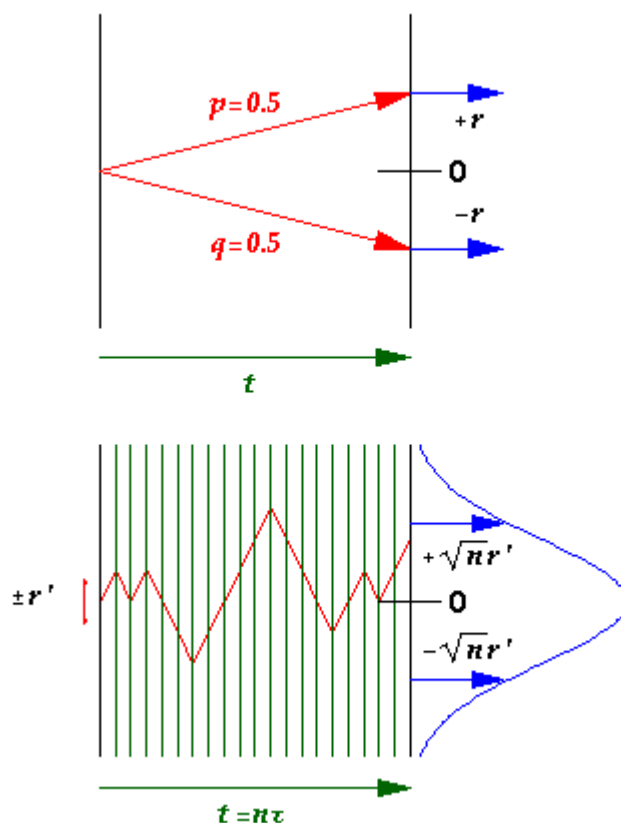
So we see that the number of forward and the number of backward steps each have a binomial distribution. Their combination gives the displacement in the case of our 1D random walk. For a d dimensional random walk, the displacement is a combination of 2D degrees of freedom, each binomially distributed. It can be verified that the Variance in the displacement is proportional to the number of degrees of freedom squared.

The motion we have just analyzed is called a **Simple Random Walk**. It is simple in the sense that the random walker makes discrete fixed movements of constant lengths at specified intervals.

The Continuous Time limit

A **Diffusion** process or **Brownian** motion can be modeled by a random walk with $p=q$ in the continuous limit. Here no matter how small the time interval δt , the walker makes a random movement $\delta X(t)$. On average this movement will be zero due to $p=q$. However the important property that the variance of a random walk scales linearly with time gives $\delta X(t)^2 \sim \delta t$. This is also known as a **Wiener process**.

To see this let's assume that our walker takes steps of length r between each time interval of duration t . Since $p=q$ his expected position at the next time step is his current position (see Martingale property) and the variance of his displacement is r^2 . To go to the continuous time limit we split the time interval t into n subintervals of duration τ and between each subinterval we allow for the walker to take steps of $\pm r'$ with equal likelihood. After a time t the position of the walker is found by summing the n independent, identical random variables Z . According to the Central Limit Theorem as n gets large the probability distribution of the particles' position will begin to resemble a Gaussian distribution with zero mean and variance $n r'^2$.



To prevent the walker from having an infinite or a zero variance as n goes to infinity our only possible choice for r' which also preserves the characteristics of our original random walker is to set

$$r' = \frac{r}{\sqrt{n}}$$

Hence if time gets rescaled by factor n then the space is rescaled by $\sqrt{n}$ and this preserves the physical properties of walker.

The continuous probability distribution $p(x,t)$ of being at position x at time t , given that $p(x_0, 0) = 1$ evolves according to a parabolic partial differential equation called the *Fokker-Planck* equation (article to follow). However, since $p=q$ there is no drift term and this equation reduces to the Diffusion equation:

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where D is called the diffusion constant and is given by:

$$D = \frac{\sigma_z^2}{2} = \frac{2r^2}{t} = \frac{2(r')^2}{\tau}$$

Finally, other stochastic processes whereby $\sigma_X(t) \sim t^{1/m}$ exist. These are called subdiffusive if $m > 2$, and superdiffusive if $m < 2$ (article to follow).

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