:: Risk Neutrality - Alternative Derivation of the Black-Scholes Equation ::

When asked to derive the Black-Scholes equation, one immediately begins to construct a portfolio of an option sort some Δ units of assets and perform on which you perform a delta hedge as originally proposed by Fischer Black and Myron Scholes. This is the the simplest way to go about it. There is, however, an alternative risk neutrality argument to deriving the Black-Scholes equation that was put forward by *Cox* and *Ross* in 1976 which does not involve delta hedging. Here we shall explore exactly this argument and make a direct comparison to the delta-hedging technique.

The concept of risk neutrality is one associated with an investment that has zero risk to asset price movement which must therefore, due to arbitrage consideration, earn the same rate as the risk free return (e.g. a bond).

The pricing dynamics of the underlying asset can be described by a geometric random walk of the form

$$\frac{dS}{S} = \mu dt + \sigma dX.$$

Further to this, we know that our call option C satisfies Ito's lemma

$$dC = \left(\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \frac{\partial C}{\partial S} dX$$

and we wish to express this in terms of a geometric random walk for the option as

$$\frac{dC}{C} = \mu_c \, dt + \sigma_c \, dX,$$

where μ_c and σ_c are the means and standard deviations of the call option respectively. Therefore,

$$\mu_c = \frac{1}{C} \left(\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right)$$

and

$$\sigma_c = \frac{\sigma S}{C} \frac{\partial C}{\partial S}$$

in order to satisfy this requirement. Rearranging our expression for μ_c gives us

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \mu S \frac{\partial C}{\partial S} - \mu_c C = 0.$$

This resembles precisely the Black-Scholes equation *if we let* $\mu_c = \mu = r$. In many literature you will find something along the lines of "...we replace μ by r to take a risk neutral preference". This is not as straight forward as it is made to sound, infact, the process of assuming the growth parameters to be equivalent to a risk free investment is a subtle point and needs to be further expanded upon.

You can construct a portfolio consisting of options and assets that is instantaneously riskless by holding $\sigma_c C$ units of asset and short selling σS units of option with a value

$$\Pi = \sigma_c CS - \sigma CS = (\sigma_c - \sigma) CS.$$

[Notice that this is different to delta hedging when one owns an option short Δ units of asset.]

Π is now written as a function of 4 variables, 3 stochastic $(S, C \text{ and } \sigma_c)$ and time t. Therefore, $\Pi \sim \Pi(S, C, \sigma_c, t)$.

To differentiate this we require Ito's Lemma for many variables. All cross terms in involving 't' vanish to slightly simplify matters, and we expand only to the second order for all other variables except t. Therefore,

$$\begin{aligned} d\Pi = {} & \frac{\partial \Pi}{\partial t} dt + \frac{\partial \Pi}{\partial S} dS + \frac{\partial \Pi}{\partial C} dC + \\ & \frac{\partial \Pi}{\partial \sigma_c} d\sigma_c + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2} dS^2 + \frac{1}{2}\frac{\partial^2 \Pi}{\partial C^2} dC^2 + \\ & \frac{1}{2}\frac{\partial^2 \Pi}{\partial \sigma_c^2} d\sigma_c^2 + \frac{\partial^2 \Pi}{\partial S\, \partial C} dS\, dC + \\ & \frac{\partial^2 \Pi}{\partial S\, \partial \sigma_c} dS\, d\sigma_c + \frac{\partial^2 \Pi}{\partial C\, \partial \sigma_c} dC\, d\sigma_c. \end{aligned}$$

After some very lengthy and tedious algebra this reduces to the much shorter expression

$$d\Pi = (\mu_c \sigma_c - \mu \sigma) CS\, dt.$$

From simple arbitrage consideration this must earn the same as a riskless interest rate $d\Pi = r\Pi\, dt$ since the structure of the portfolio is such that the risk is eliminated. Using this and our expressions above we arrive at the expression

$$\mu_c \sigma_c - \mu \sigma = r(\sigma_c - \sigma),$$

Alternatively written as

$$\frac{\mu_c - r}{\sigma_c} = \frac{\mu - r}{\sigma}.$$

The interpreting of this equation has a great financial significance. It says that the ratio extra rate return over a risk free investment of option and asset with their respective volatilities is fixed. This ratio is often termed the *market price of risk* which has already been mentioned in the article

'Log-Normality & Finance'. Here we have shown that an option and the underlying asset have the same ratio within a *risk neutral world* framework. From this equation we can interpret what we already know - the bigger the returns the greater the risk.

By using the expression of market price of risk and substituting it into our expression for μ_c and σ_c above we recover the Black-Scholes equation

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0,$$

So we have not simply taken equation above with $\mu_c = \mu = r$, but done something more subtle This choice means that this ratio of market price of risk can be satisfied for any set of values for σ_c and σ. This is what makes the Black-Scholes model attractive. Furthermore, it sets a simple yet well defined 'universal standard' as desired like in any field. By letting $\mu = r$ is different to saying that in reality no investment can grow faster than the rate r, but simply to set a fair price for our derivative we must let the two be equivalent.

Here we have derived the Black-Scholes equation without performing a delta hedge as is most often presented in common literature. Delta-hedging is a more refined and sophisticated extension to the risk neutrality argument, yet has the simplicity of creating a portfolio with just long and option and short some divisible unit of assets which makes it so attractive for practical application (also having also two less parameter to worry about).

One has to ask themselves where would the derivatives market be today if there was not some widely agreed model to go by? Afterall, alternative derivative pricing methods did exist pre-Black-Scholes.

Written by Alessio Farhadi. Vote of thanks to Myron Scholes.

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