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FUNDAMENTALS

:: Basic Probability Theory 1 - Set Theory ::

This article is intended as an introduction to basic set theory for those aiming to apply it in the field of probability theory and (eventually) stochastic calculus. Most people would have encountered set theory in some context, even without knowing it. A set is merely a means of organizing data. It is denoted (mathematically) by the curly brackets: '{ }'. A set collects together data that are related in some way. This sounds ambiguous (how do you define 'related'), because it is ambiguous! A set person looking at the following set:

$$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

can argue that this is a logical set, being the set of symbols from which any number can be constructed. Another set:

$$B = \{15.65, 14.42, 16.87, 14.97, 15.03\}$$

may appear to follow no logic, but it may be the set of times in seconds that a person has taken to complete a 100 meter trials in his/her last 5 heats. Both are sets in the sense that they put together data that is related in some way. Each number in each of the sets is an element or member of the set. This is denoted by the symbol ' $\in$ ', for e.g. $15.65 \in B$, read '15.65 is a member (or element) of set B'.

The following definitions will introduce some of the key manipulations used in set theory. Their relevance and uses will appear limited for now, but they are used extensively in probability theory and stochastic processes. Hence, the following should be read as a set of 'rules' for the moment. The clarification will follow in a later article which uses the concepts below in discussing real probability problems.

Definition 1:

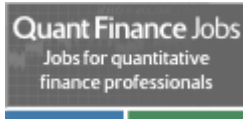
Union - Denotes the amalgamation of all the elements of 2 or more sets. It is symbolized by the symbol ' $\cup$ '.

Example 1: The union of the sets A and B defined above is given by:

$$A \cup B = \{0, 1, 3, 4, 5, 6, 7, 8, 9, 15.65, 14.42, 16.87, 14.97, 15.03\}$$

With an additional set $C = \{100, 200\}$,

$$A \cup B \cup C = \{0, 1, 3, 4, 5, 6, 7, 8, 9, 15.65, 14.42, 16.87, 14.97, 15.03, 100, 200\}$$



:: Collaborations ::



In basic probability the probability of the union of 2 events A and B is often described by the probability of events A or B being observed. This idea extends to this definition (in fact, it is the other way around; the use in probability extended from set theory), where the union of A and B denotes those elements that are in A or B.

Hence, the union of the sets A_i for $i = 1, 2, 3 \dots n$ is denoted by:

$$\bigcup_{i=1}^n A_i$$

and this amalgamates all the elements from the individual sets A_i into a new set.

Definition 2:

Intersection - Denotes the set of elements that are members of all the sets under consideration. The symbol used is ' $\cap$ '.

Example 2: Let $C = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $D = \{2, 4, 6, 8, 9\}$ be two given sets (the significance of their elements is not of concern). The intersection of the two sets is also a set that contains the elements that are common to both sets C and D.

$$C \cap D = \{2, 4, 6, 8\}$$

since these elements are present in both sets. Again, turning to the more familiar case in probability the intersection of 2 events C and D is described as the event that both C and D occur. As another example consider the intersection of sets A and B defined above. This does not have any elements, since there are no elements that are common to both A and B. The terminology adopted to denote such a set is the empty set. It is symbolized by ϕ .

$$A \cap B = \phi$$

Similarly, the intersection of the set of even integers with the set of odd integers is also an empty set, and the intersection of a number of sets B_i for $i = 1, 2 \dots n$ is:

$$\bigcap_{i=1}^n A_i$$

and this operation creates a set of elements that are common to all B_i .

Definition 3:

Subset - A set that contains some or all of the elements of another set, but nothing more. The part in *italics* is key. The set $\{1, 3, 5, 7\}$ is a subset of the much larger set of all odd integers but it is not a subset of the set $\{1, 3, 5\}$, since 7 is not a member of the smaller set.

The symbol for subset is ' $\subset$ '. Hence, $\{1, 3, 5\} \subset \{1, 3, 5, 7\}$

The idea of a 'super-set' will also be encountered in literature ($\supset$). This has an opposite meaning to subset. A set A is a super set of another set B if A encompasses all of the elements of B. Hence, for A to be a super-set of B, B must be a subset of A.

$$\{1, 3, 5, 7\} \supset \{1, 3, 5\}$$

Definition 4:

Power set - A power set of a set X is the set of all subsets of X . The best way to proceed is via example:

Example 3: Let $X = \{1, 3, 5\}$. Then each of the following are subsets of X - $\{1, 3, 5\}$, $\{1, 3\}$, $\{1, 5\}$, $\{3, 5\}$, $\{1\}$, $\{3\}$ and $\{5\}$.

Note: $\{1, 3\} = \{3, 1\}$ since it contains the same elements. The ordering is irrelevant in set theory since the only point of interest is the data that a set holds.

The power set is thus denoted by $P = \{ \{1\}, \{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, X \}$. This construction of a set is slightly different from the portrait of a set given so far in this article. However, the idea remains the same. The power set P is still a set, in that it contains elements of data that are related somehow. The elements $\{1\}$, $\{3\}$ and $\{5\}$ each hold the same meaning as they do in the set X , but the 'dual' sets $\{1, 3\}$, $\{1, 5\}$ and $\{3, 5\}$ hold something extra. From the notation introduced earlier, these are in fact unions. The union of $\{1\}$ and $\{3\}$ is $\{1, 3\}$. Similarly the union of $\{1\}$, $\{3\}$ and $\{5\}$ is X . The union of $\{1, 3\}$ and $\{3\}$ is $\{1, 3\}$ and so on. Hence, the elements are now more complex. The elements of P can themselves appear like sets! They are however, not sets. They are still elements in the sense that $\{1, 5\}$ or e.g. is still one 'item' and not two even though it contains 2 numbers.

Definition 5:

Complement - The complement of a subset M (of a larger set N) is simply the set that is not part of set M , but still part of N . It is denoted by a 'c' in the superscript.

Example 4:

Using set X as defined in the previous example, the complement of $\{1\}$, also denoted by $\boxed{\complement}$ is simply the elements left over from X when the element $\{1\}$ is removed.

$$\{1\}^c = X - \{1\} = \{3, 5\}$$

If the set under consideration is P (the power set from the example above), then

$$\{1\}^c = P - \{1\} = \{\{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, X\}$$

It follows that the complement of a set is the empty set (though such a concept is of limited practical use):

$$\begin{aligned} X^c &= X - X = \phi \\ P^c &= P - P = \phi \end{aligned}$$

Definition 5:

Field (algebra) - A set F is called a field (or algebra) if its members satisfy the following requirements:

1/ $\phi \in F$ (i.e. the empty set is included in F)

2/ If $A \in F$ then $A^c \in F$ (this holds for any elements of F , whether these elements are trivial, single numbers, or whether they are smaller sets

such as those encountered in the power set).

3/ If $A, B \in F$ then $A \cup B \in F$ (again this holds for any element of F).

This can be extended to more unions so that if $A_i, i = 1, 2, \dots, n$ are elements of F then:

$$\bigcup_{i=1}^n A_i \in F$$

None of the sets encountered so far are fields. The closest candidate is the power set in example 3, but since this set excluded the empty set it does not qualify to be classified as a field. Fields can be (and usually are) very complicated due to the constraints on the type of elements that they can hold. This is demonstrated in the following example. It is very rare to encounter fields occurring naturally. They are usually created to deal with certain problems; and in creating a field, there must be an initial set from which the field is created (or in correct terminology generated).

Example 5:

Consider the field (set F) generated by the set $X = \{1, 3, 5\}$.

- (i) $\emptyset \in F$ by rule 1.
- (ii) The field generated by X implies $\{1\} \in F, \{3\} \in F$ and $\{5\} \in F$.
- (iii) From rule 2, the complements of each element in X must be included. Thus the elements $\{1, 3\}, \{1, 5\}$ and $\{3, 5\}$ are members of F as they correspond to $\{5\}^c, \{3\}^c$ and $\{1\}^c$ respectively.
- (iv) From rule 3, the union of all possible combinations of the elements of F must in turn be members of F :

So far,

$$F = \{\emptyset, \{1\}, \{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}\}$$

The only 'new' element that can be constructed by taking unions is the following:

$$\{1, 3\} \cup \{1, 5\} \cup \{3, 5\} = X$$

and

$$F = \{\emptyset, \{1\}, \{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, X\}$$

This completes the exercise. Had the initial set X involved just one more term, the field would look a lot more complicated. For instance, the field generated by the set $Y = \{1, 3, 5, 7\}$ is:

$$F_Y = \{\emptyset, \{1\}, \{3\}, \{5\}, \{7\}, \{1, 3\}, \{1, 5\}, \{1, 7\}, \{3, 5\}, \{3, 7\}, \{5, 7\}, \{1, 3, 5\}, \{1, 3, 7\}, \{1, 5, 7\}, \{3, 5, 7\}, Y\}$$

Definition 6:

Measurable - A set X is said to be measurable with respect to a field F if every element of X is a member of F .

As a consequence of the definition, one may conclude that X in example 5 is measurable with respect to both F and F_Y . Y on the other hand is only measurable with respect to $\square$. This shows that the field generated by a set is the smallest field on which it is measurable. Such fields will be of interest in probability theory.

Fields are a key theme in probability theory. The data they contain (and more importantly the structure of this data as a result of the requirements of a field) is used to introduce the concept of events in probability problems.

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