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:: FUNDAMENTALS

:: Solving the Black-Scholes Equation ::

We have the Black-Scholes equation for European non-dividend-paying call option

$$\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0,$$

with boundary conditions

$$C(0, t) = 0, \quad C(S, t) \sim S \text{ as } S \rightarrow \infty,$$

and

$$C(S, T) = \max(S - E, 0).$$

These conditions for a European call option with expiry T and exercise price E arise from: an option being worthless if the price of the underlying security is zero itself; the price of an option C(t) cannot be greater than the price of the underlying security S(t) otherwise an arbitrage can be achieved by buying the security and selling the call option.

In fact, $C(t) \leq S(t)$, i.e. the maximum value of the call option is the difference between the spot price and the exercise price.

The Black-Scholes equation resembles a linear parabolic equation similar to the diffusion equation. A general linear parabolic equation is of the form:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial X^2} + a \frac{\partial u}{\partial X} + bu$$

where a and b are constants. This can always be reduced to a diffusion equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial X^2}$$

by choosing the substitution

$$v = u l(x) g(t)$$

We find that this condition can be satisfied by setting

$$l(x) = \exp(ax/2)$$

and

$$g(t) = \exp\left[\left(\frac{a^2}{4} - b\right)t\right]$$

If we assume r and σ to be constant, the Black-Scholes equation can be

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reduced to the diffusion equation. To make this possible we set

$$S = E \exp(x), \quad \tau = T - t / \left(\frac{1}{2} \sigma^2 \right),$$

$$C = E v(x, \tau)$$

The result is

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1) \frac{\partial v}{\partial x} - kv$$

where

$$k = r / \left(\frac{1}{2} \sigma^2 \right)$$

Having used this substitution, our initial condition becomes

$$v(x, 0) = \max(e^x - 1, 0)$$

Now our equation looks more like the diffusion equation and we can obtain the similarity solution using

$$v = \exp(\alpha x + \beta \tau) u(x, \tau)$$

By substituting this expression for v into the parabolic equation above and choosing values of α and β such that all terms involving $u(x, \tau)$ are eliminated, we obtain

$$v = \exp\left(-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2 \tau\right) u(x, \tau)$$

with

$$u(x, 0) = \max\left(\exp\left(\frac{1}{2}(k+1)x\right) - \exp\left(\frac{1}{2}(k-1)x\right), 0\right)$$

For now it is just a matter of solving the diffusion equation by methods of **Fourier Transform**.

The final solution is therefore

$$C(S, t) = S N(d_1) - E e^{-r(T-t)} N(d_2)$$

where

$$d_1 = \frac{\log(S/E) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}$$

and

$$d_2 = \frac{\log(S/E) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}$$

Here $N(\cdot)$ is the Cumulative Normal Distribution Function, the exact form is given by:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}s^2} ds$$

A simpler polynomial approximation up to 6 decimal places accurate can be used to represent this function instead:

$$N(x) = 1 - N'(x) (a_1 k + a_2 k^2 + a_3 k^3 + a_4 k^4 + a_5 k^5) \text{ when } x \geq 0$$

and

$$N(x) = 1 - N(-x) \text{ when } x < 0$$

Here

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$k = \frac{1}{1 + \gamma x}$$

$$\gamma = 0.2316419$$

$$a_1 = 0.319381530$$

$$a_2 = -0.356563782$$

$$a_3 = 1.781477937$$

$$a_4 = -1.821255978$$

$$a_5 = 1.330274429$$

This is the exact solution for calculating the value of an European non-dividend paying call option. For put option of the same expiry T and exercise price E, the **put-call parity** formula can be used if the value of the call option is first calculated

$$C - P = S - E e^{-r(T-t)}$$

This gives value of P as

$$P(S, t) = E e^{-r(T-t)} N(-d_2) - S N(-d_1)$$

Written by Alessio Farhadi.

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