



:: Stochastic Integration ::

In science, random processes are often described by a **Langevin Equation**:

$$\frac{dS}{dt} = a(S,t) + b(S,t)\,\xi(t),$$

where S is the variable of interest, a(S,t) and b(S,t) are certain known functions and $\xi(t)$ is a Gaussian fluctuation with the following properties:

[Graphics:Images/stochinteg

It can be shown by using the above properties that

$$\int_0^T \xi(t)\,dt = X(T).$$

We see that the integral in time of $\xi(t)$ is X(t), the **Wiener process**, which itself is not differentiable. Therefore, mathematically speaking only the integral equation of the Langevin equation is defined. From this we use the following interpretation

$$dX(t) \equiv X(t+dt) - X(t) = \xi(t)\,dt.$$

Let G(t) be an arbitrary stochastic function of t and X(t).We shall consider the stochastic integral

$$\int_{t_0}^{t_n} G(s)\,dX(s).$$

where $t_0 \le t_1 \ldots \le t_n$ and $t_{i-1} \le \tau_i \le t_i$ for all i up to $t_i = t_n$. The integral above is to be interpreted as a kind of *Riemann-Stieltjes* integral which can therefore be interpreted by the following limiting discretized sum

$$S_n = \sum_{i=1}^{n} G(\tau_i)\,\{X(t_i) - X(t_{i-1})\}$$

$$dt = t_i - t_{i-1}$$

Consider $G(\tau_i) = X(\tau_i)$. Then

$$E_{t_n}[S_n] = E_{t_n}\left[\sum_{i=1}^{n} X(\tau_i)\,\{X(t_i) - X(t_{i-1})\}\right]$$

$$= \sum_{i=1}^{n} \{\min(\tau_i, t_i) - \min(\tau_i, t_{i-1})\}$$

:: Collaborations ::

$$= \sum_{i=1}^{n} \tau_i - t_{i-1}$$

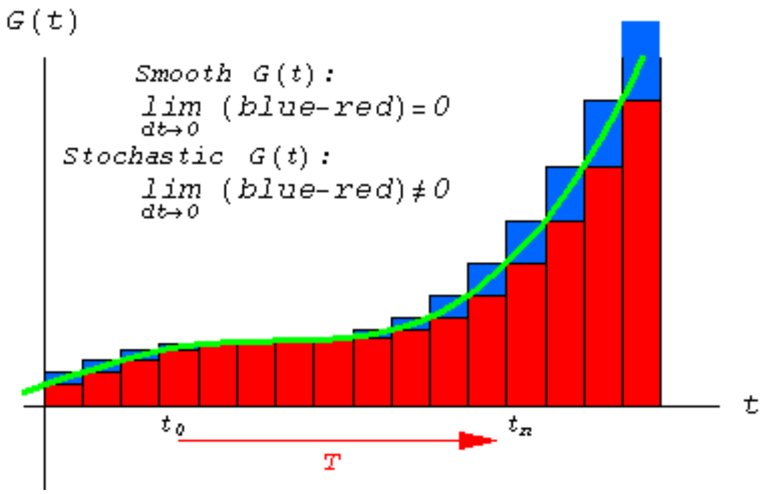
Since $t_{i-1} \le \tau_i \le t_i$ we can write it as $\tau_i = \alpha t_i + (1-\alpha)\, t_{i-1}$ with $0 \le \alpha \le 1$. Substituting this in the sum above we find that

$$E_{t_n}[S_n] = \alpha\,(t_n - t_0)$$

This means that the outcome of our integration depends on where in each interval we evaluate G(t). This is not so if G(t) were a smooth function with no stochastic dependence. Indeed, if this were the case then taking $\tau_i = t_{i-1}$ (red) or $\tau_i = t_i$ (blue) would produce the same value for the integral as dt->0. Instead, for $G(\tau_i) = X(\tau_i)$ we have

$$\int_{\tau_i = t_i} X(s)\, dX(s) - \int_{\tau_i = t_{i-1}} X(s)\, dX(s) \to t_n - t_0 =$$

$$T \quad [*]$$



We will use the Ito stochastic integral where [image] r so that $\tau_i = t_{i-1}$. There is a good reason for this: in all natural cases unknown future events cannot affect the present. This means that the function G(t) is non-anticipating in that it cannot be used to predict the future increment in dX. This is of course equivalent in saying G(t) is a martingale since what we mean is exactly that

$$E_s[G(t)] = G(s) \text{ for } s \le t.$$

We only know what is the present value of G(t) which corresponds to that at the beginning integrating interval. For this reason it is more appropriate to use the Ito integral. It is important also to note that integrating a non-anticipative function with respect to dt or dX is itself non-anticipating. So for G(t) =X(t) the Ito integral becomes:

$$\int_0^T X(s)\, dX(s) \approx \sum_{i=1}^{n} X_{i-1}\,(X_i - X_{i-1})$$

$$I = \sum_{i=1}^{n} X_i\,(X_i - X_{i-1}) - \sum_{i=1}^{n} (X_i - X_{i-1})^2$$

$$= \sum_{i=1}^{n} (X_i^2 - X_{i-1}^2) - I - \sum_{i=1}^{n} (X_i - X_{i-1})^2$$

$$\Longrightarrow 2\ I = X\ (T)^2 - T$$

$$\int_0^T X\ (s)\ dX\ (s) = \frac{X\ (T)^2 - T}{2}$$

Notice that $2 \sum_{i=1}^{n} X_i\ (X_i - X_{i-1}) = X\ (T)^2 + T$ and hence the result in Eq. [*] follows.

So what does this integral mean?

If we were to run many realizations of the Wiener processes in the interval [0,T] and compute the sum of X(s) dX(s) for each, the average sum would be zero. This is because $\langle X_i \Delta X_i \rangle$ vanish since X_{i-1} is statistically independent from the increment [Gr. Instead, the stochastic has to be interpreted as how the Variance of the processes changes with time. It is therefore a mean square limit of sum of the discretized process.

[Graphics:Images/stochintegration_gr

$$= E_T \left(\int_0^T (G\ (t))^2\ (dW)^2 \right)$$

We can use this to compute other integrals:

$$\int_0^T X\ (t)\ dt = E_T \left(\int_0^T \int_0^T X\ (t)\ X\ (s)\ dt\ ds \right)$$

$$= \int_0^T \int_0^T E_T\ (X\ (t)\ X\ (s))\ dt\ ds$$

$$= \int_0^T \int_0^T \min\ (t,\ s)\ dt\ ds$$

$$= \int_0^T \left(\int_0^s t\ dt + \int_s^T s\ dt \right) ds = T^3 / 3$$

The integral of a polynomial in X(t):

$$d[X\ (t)]^n = [X\ (t) + dX\ (t)]^n - X\ (t)^n$$

$$= \sum_{r=1}^{n} \binom{n}{r} X\ (t)^{n-r}\ dX^r$$

$$= n X\ (t)^{n-1}\ dX\ (t) + \frac{n\ (n-1)}{2} X\ (t)^{n-2}\ dt$$

(by the quadratic variation of X(t))

$$\Rightarrow \int_0^T X(t)^n \, dX = \frac{X(t)^{n+1}}{(n+1)} - \frac{n}{2} \int_0^T X(t)^{n-1} \, dt$$

Note: the integral of G(t) with respect to dX and dt are two different things and, generally, have no connection.

Written by Raffaello Vardavas.

Back for more