Quant Notes





:: Collaborations ::

:: Stochastic Integration & Ito's Lemma ::

This article combines the previous article on the subject to **Ito's Lemma**.

We have seen that:

$$\int_0^T X(s)\, dX(s) = \frac{X(T)^2 - T}{2}$$

By Ito's Lemma we have that:

$$\int_0^T dF = \int_0^T \frac{\partial F}{\partial t} dt + \int_0^T \frac{\partial F}{\partial X} dX + \frac{1}{2}\int_0^T \frac{\partial^2 F}{\partial X^2} dt$$

Therefore if we set $\frac{\partial F}{\partial X} = X$ thus $F = X^2/2$ then on rearranging we have:

$$\int_0^T \frac{\partial F}{\partial X} dX = \int_0^T X(s)\, dX(s)$$

$$= \int_0^T dF - \int_0^T \frac{\partial F}{\partial t} dt - \frac{1}{2}\int_0^T \frac{\partial^2 F}{\partial X^2} dt$$

$$= \frac{X^2}{2} - 0 - \frac{T}{2} = \frac{X(T)^2 - T}{2}$$

e.g. Show that

$$\int_0^T X(t)^2\, dX(t) = \frac{X(T)^3}{3} - \int_0^T X(t)\, dt$$

set $\frac{\partial F}{\partial X} = X^2$ thus $F = X^3/3$.

$$\int_0^T \frac{\partial F}{\partial X} dX = \int_0^T X(t)^2\, dX(t)$$

$$= \frac{X(T)^3}{3} - 0 - \int_0^T X(t)\, dt$$

e.g. Show that

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set $\frac{\partial F}{\partial X} = t$ thus $dF = d(X(t)\,t)$.

$$T X(T) = \int_0^T \frac{\partial F}{\partial t} dt + \int_0^T \frac{\partial F}{\partial X} dX + \frac{1}{2}\int_0^T \frac{\partial^2 F}{\partial X^2} dt$$

$$= \int_0^T X(t)\, dt + \int_0^T t\, dX(t) + 0$$

Solving a Stochastic Differential Equation

Consider the linear multiplicative noise process :

$$dS = \sigma S\, dX(t)$$

This process is known as multiplicative white noise since the noise term dX multiplies S. This can be solved using Ito's formula:

$$F(S, t) = \int \frac{dS}{S} = \log S$$

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$$= 0 + \frac{1}{S} dS - \frac{1}{2 S^2} dS^2$$

$$= \sigma dX - \frac{1}{2 S^2} (\sigma S)^2 dX^2 = \sigma dX - \frac{\sigma^2}{2} dt$$

We can now integrate directly:

$$S(t) =$$

$$S(t_0) \operatorname{Exp}\left\{\sigma [X(t) - X(t_0)] - \frac{\sigma^2}{2} (t - t_0)\right\}$$

Written by Raffaello Vardavas.

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