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:: FUNDAMENTALS

:: Volatility ::

The Black-Scholes equation requires the knowledge of the spot rate r , the value of the underlying $S(t)$ at time t (which is known at the current time) and most of all the value of the volatility of the underlying σ . Out of the above, the volatility is the most difficult estimate.

One method is to use past data on the volatility of the underlying to estimate future ones. Empirically it has been observed that the best estimates are given by considering historical data in an interval Δt , long as the maturity time T of the option.

A better approach is to estimate it directly from the market option quotes. This implied volatility σ_{imp} is calculated numerically given that the option value $C(S, t)$ is

$$C(S, t) = S N(d_1) - E e^{-r(T-t)} N(d_2)$$

where

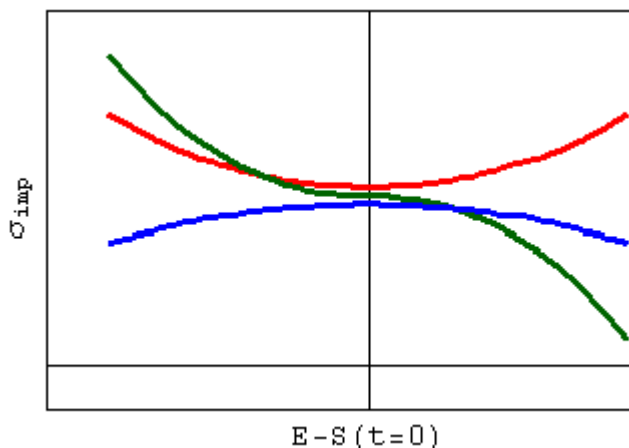
$$d_1 = \frac{\log(S/E) + (r + \sigma_{imp}^2/2)(T-t)}{\sigma_{imp} \sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma_{imp} \sqrt{T-t}$$

Generally the implied volatility gives a better estimate than that obtained by historical data.

The implied volatility is found to be a function of both strike price and maturity. The longer the time to maturity, the larger the risk taken and thus the larger the implied volatility.



The most common dependence with strike price is given by a volatility

smile (red curve in the figure above), where the volatility is minimum when the strike price is given by the initial value of the underlying. Other dependences such as volatility **frowns** (blue curve) and **smirks** (green) are also observed.

Another approach of dealing with volatility is to assume that it follows a Stochastic Differential Equation (SDEs). We can follow a type of Black Scholes derivation (see article). So we have our usual log normal SDE and one other for the volatility.

$$\begin{aligned} dS &= \mu S dt + \sigma S dX_1 \\ d\sigma &= p(S, \sigma, t) dt + q(S, \sigma, t) dX_2 \end{aligned}$$

The two Brownian motions are correlated such that $dX_1 dX_2 = \rho dt$, where ρ is a measures the correlation. Assuming that the functions p and q are given, how do we form a riskless portfolio? We need to hedge our option $V(S, t)$. Since we cannot hedge against σ , because it is not offered in the market, we need to hedge against another option $\tilde{V}(S, t)$ on the same underlying S . So as usual we create our hedging portfolio Π .

$$\Pi = V - \Delta S - \tilde{\Delta} \tilde{V}$$

By Ito:

$$\begin{aligned} dV &= V_t dt + V_S dS + V_{SS} dS^2 / 2 \\ &\quad + V_{\sigma} d\sigma + V_{\sigma\sigma} d\sigma^2 + V_{S\sigma} dS d\sigma \end{aligned}$$

where our notation V_t denotes a partial derivative of V w.r.t. to time...,etc. From the SDEs we have that

$$\begin{aligned} dS^2 &= \sigma^2 S^2 dt \\ d\sigma^2 &= q^2 dt \\ dS d\sigma &= \rho q dt \end{aligned}$$

We substitute these relations into the above expansion for dV and a similar one for $d\tilde{V}$. Substituting these into the expression for the change in portfolio value, given by

$$d\Pi = dV - \Delta dS - \tilde{\Delta} d\tilde{V}$$

we find that to remove the randomness of the Brownian motion we need to choose

$$\begin{aligned} \tilde{\Delta} \tilde{V}_{\sigma} &= V_{\sigma} \\ V_S - \Delta - \tilde{\Delta} \tilde{V}_S &= 0 \end{aligned}$$

which gives

$$\Delta = V_S (1 - V_{\sigma} / \tilde{V}_{\sigma})$$

Now we can use the no arbitrage condition to equate  to the riskless return of investing Π in a bank at a fixed rate r .

 [Graphics: Images/vola](#)

We are then left with a risk neutral partial differential equation (PDE) involving partial derivatives of V and $\tilde{V}$. Defining the differential operator

$\mathcal{D}$ such that

$$\mathcal{D}f = \frac{f_t + \sigma^2 S^2 f_{SS} / 2 + \rho \sigma q S f_{S\sigma} + q^2 f_{\sigma\sigma} / 2 - r f}{f_\sigma}$$

the final PDE can be expressed as follows:

$$\mathcal{D}V = \mathcal{D}\tilde{V}$$

The left hand side is a function of V but not of $\tilde{V}$, and vice versa for the right hand side. This means that since the two options will in general have different strike price and maturity dates, this equality holds if both sides are independent of the contract specifications. We can thus equate this to a function of the independent quantities S , σ and t .

For some arbitrary function $\lambda(S, \sigma, t)$ we have that

$$\mathcal{D}V = -(p - \lambda q)$$

$p - \lambda q$ is called the **risk neutral drift rate** of volatility, whilst the function λ is called the **market price of risk volatility** (articles to follow).

Written by Raffaello Vardavas.

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