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:: The Wiener Process ::

Following on from the article on the **Simple Random Walk**, we can build a continous random walk path from the variable X_n by interpolating linearly bewteen different points:

$$Y(t) = X_i + (t - n\,dt)(X_{i+1} - X_i)$$

where i $dt \le t \le (i+1)dt$. These paths have the following properties:
1. Markov Property: If $T = n\,dt$ and $\tau > 0$, the increment $Y(T+\tau) - Y(T)$ is independent of the "history" $\{Y(t),\ t \le T\}$;
2. $E(Y(T)) = 0$.
3. [Graphics].

The increment in Y (T) is a sum of many independent binomial random variables.Therefore if we define the Wiener process to be $X(t) = \lim_{dt\to 0} Y(t)$ with X(0)=0 then by the **Central Limit Theorem** the probability distribution of the increment X(t+a)-X(t) is Gaussian with zero mean and variance a. Furthermore, by the Markov property, this increment is independent of X(t) for $0 \le t \le T$. Since variance of the sum of independent quantities is equal to the sum of the individual valances, it follows that:

$$1.\ \int_0^T dX(t)^2 = \left\langle \lim_{dt\to 0} \left\{ \sum_{i=1}^{n} (X_i - X_{i-1})^2 \right\} \right\rangle$$

$$= \left\langle \lim_{dt\to 0} (X_n)^2 \right\rangle = T$$

$$= \int_0^T dt \implies dX(t)^2 = dt$$

$$2.\ E_t[X(t+a) - X(t)] = 0.$$

where E_t denotes an expectation operator at time t. Property (1) is known as the quadratic variation of the *Wiener path* on the interval $[0, T]$. Property (2) instead is the Martingal or fair game property since it means that we cannot anticipate the value of $X(t+a)$ at time t no matter how close in the future t+a is:

$$E_t[X(t+a)] = E_t[X(t)].$$

So for $s \le t$, it follows that

$$E_s[X(t)] = E_s[E_t[X(t)]],$$

$$E_s[X(s)\,X(t)] = \min(t, s) = s.$$

It is due to the finiteness of the quadratic variation of the **Wiener process** that implies that its paths are not differentiable. Indeed, for any differentiable function, as $dt \to 0$ then $X_{i-1} \to X_i$. Property (1) also implies that the Wiener process is statistically self-similar:

$$X(t) \mapsto \lambda^{-1/2} X(\lambda t)$$

A generalization of the quadratic variation of the Wiener process is that

$$\Longrightarrow dX(t)^N = \begin{cases} dt & \text{for } N = 2 \\ 0 & \text{for } N > 2 \end{cases}$$

This comes from the fact that a Gaussian distribution is described just by the first two central moments.

Written by Raffaello Vardavas.

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